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Properties of primes and natural mathematics *A minimalist algorithm for prime numbers***

ABSTRACT

This paper discusses precise quantification by means of number systems on the analogy of Jaspers' (2005) earlier analysis of the comparatively vague type of quantification expressed by predicate calculus operators {all/every/each, some, no}. It is argued that numbers provide an interesting testing ground for the validity of the Boolean approach to quantifiers in Jaspers (2005). More specifically, this excursion into maths is undertaken to show that a very basic cognitive- logical system of oppositions which underlies natural language logic governs natural mathematics as well. The concrete starting point of the article is Popper's twin prime problem, which is followed by a discussion of number systems, more specifically the distinction between the natural number system {(0,) 1, 2,...} and the prime number system. The former type of system will be argued to be orders characterized by the operation of addition/subtraction. The prime number sequence is different in that it is multiplicative/divisional rather than additive. It is generally recognized in mathematical circles that the latter type of sequence is more complex than the former. This fact tallies well with (and hence provides indirect support for) the linguistic findings in Jaspers (2005), whose core was the claim that natural language disjunction – known to be isomorphic with addition in algebra – is cognitively and lexically less complex than conjunction, which is isomorphic with multiplication.

Keywords: Quantification; numerical systems; system of oppositions; disjunction; conjunction.

RESUMO

Este trabalho discute a quantificação precisa por meios de sistemas numéricos em analogia à análise anterior de Jaspers (2005) a respeito da quantificação comparativamente vaga expressa por operadores do cálculo de predicados {todos, todo, cada, algum, nenhum}. É defendido que números oferecem um interessante teste-base para a validade da abordagem Booleana aos quantificadores (Jaspers, 2005). Mais detidamente, esta excursão na matemática é realizada para mostrar que o mesmo sistema lógico-cognitivo de oposições subjacente na língua natural também governa a matemática natural. O ponto de partida concreto do artigo é o problema dos "twin primes" de Popper, que é seguido por uma discussao de sistemas de números, sobretudo a distinção entre sistema dos numeros naturais {(0,) 1, 2,...} e o sistema de números primos. Em relação ao primeiro será defendido que é organizado pela operação de adição/subtração. A sequência de números primos é diferente, porque é mais multiplicativa/divisional que aditiva. É geralmente reconhecido em círculos matemáticos que o último tipo de sequência é mais complexo que o primeiro. Este fato acompanha bem (e, portanto, oferece suporte indireto para) as descobertas linguísticas em Jaspers (2005), cujo o núcleo foi a defesa que disjunção na língua natural _ conhecida por ser isomórfica à adicao na álgebra álgebra - é cognitiva e lexicalmente mais complexa que a conjunção, que é isomórfica à multiplicação.

Palavras-chave: Quantificação; sistemas numéricos; sistema de oposições; disjunção; conjunção.

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Introduction

In this paper we shall discuss *precise* quantification by means of number systems on the analogy of our earlier analysis (JASPERS, 2005) of the comparatively vague type of quantification expressed by predicate calculus operators {all/every/each, some, no(ne)}. Number theory represents one of science's most fruitful *inventios* and consequently an interesting testing ground for the validity of the Boolean approach to quantifiers in Jaspers (2005). More specifically, this excursion into maths is undertaken to show that a very basic cognitive-logical system of oppositions which underlies natural language logic governs natural mathematics as well. In the first section of the paper, the notion of twin primes is introduced. The discussion of number systems will then start by distinguishing between the natural number system {(0,) 1, 2,…} and the prime number system. Infinite number sequences of the former type will be argued to be orders characterized by the operation of *addition*. The prime number sequence is different in that it is *multiplicative* rather than additive. It is generally recognized in mathematical circles that the latter type of sequence is *more complex* than the former. This fact tallies well with (and hence provides indirect support for) independent linguistic findings in Jaspers (2005), whose core was the claim that natural language disjunction – known to be isomorphic with addition in algebra – is cognitively and lexically less complex than conjunction, which is isomorphic with multiplication.

1 Twin Primes

Consider the following problem described in Popper's (1994) *Knowledge and the body-mind problem.*

> Similarly, as I said last time in the discussion, we may invent a method of naming the natural numbers so that we can, in principle, always add one, and so go on to infinity. This is our invention, in this case belonging to the Babylonians. But from this invention there emerge unintended and unavoidable consequences which we neither invent, nor make, but discover. For example, that there are odd and even numbers; or that there are divisible numbers and prime numbers such as 2, 3, 5, 7, 11, 13, 17, 19, 23, 29 and 31. These prime numbers have given rise to many solved and many more yet unsolved problems. For example, the problem 'Does the sequence of prime numbers fizzle out or do they go on for ever?' has been solved by Euclid. Although they occur less and less often as we go along, they never fizzle out: there is no end to them. Euclid's proof is very simple and very beautiful, but I do not have enough time to state it here. There are lots of unsolved problems, for example 'Do twin primes fizzle out?' (Twin primes are primes with exactly one even number between them, such as 3 & 5; 5 & 7; 11 & 13; 17 & 19; 29 & 31. They are called twin primes because they are very close to each other, as close as two primes can possibly be to each other.) Now the question whether or not the twin primes fizzle out is one of the unsolved problems of number theory. We just do not know. Popper (1994: 30).

While no attempt will be made to solve the twin prime problem – which is a topic for scholars versed in mathematics, computing and in particular in higher *number theory* – we shall try to develop a highly minimalist algorithm for primes. Our main claim is that a minimal use of this algorithm may represent the basic multiplicative mathematical system human beings are innately equipped with.

That there is an association between primes and multiplication is straightforward from the definition of a prime: "a whole number greater than 1 is ***prime*** if it cannot be written as the *product* of two smaller whole numbers" (Dunham 1994: 2; italics mine). Dunham (1994, p. 3) claims that this link with multiplication is also the reason why higher number theory is very complex and difficult. The number system, so he claims, is at its root an additive system: "whole numbers are literally created by the operation of addition". Now, prime numbers, which "may lie at the heart of the higher arithmetic [...] also are responsible for its greatest mathematical snarls" because "questions about primes and composites[1] thrust *multiplication* into the system". The source of the complexity is that "mathematicians try to examine additive creations under a multiplicative light."

2 A minimalist Algorithm

Let us consider the prime properties of the first 25 numbers of the set of natural numbers:

(1)
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24

A first observation is that Popper's claim that twin primes (such as 5 and 7, 11 and 13, 17 and 19) are "as close as two primes can possibly be to each other" is not fully correct: 2 and 3 are uncontroversially viewed as primes and there is no number between them. For all primes > 3, though, Popper's claim is correct. Secondly, it is striking that the three twin primes of this set of numbers occur on either side of the number 6 or a multiple of 6, a feature which is well-known in mathematical circles but also of great importance for the development of a minimalist algorithm, as will be shown below. Finally, a remark is in order concerning the status of 1, which is generally taken not to be a prime number. At first sight, this is a strange position in that 1 fits the definition of a prime: "it cannot be written as the *product* of two smaller whole numbers" (DUNHAM 1994: 2) or – in terms of another definition of primes commonly appealed to – it is divisible both by itself and by 1. The difference with primes such as two and three, however, is that the divisibility of *one* is a

[1] "If a whole number greater than 1 is not prime – which is to say, if a number possesses an integer factor other than 1 and itself – we say it is composite. Numbers such as 24 = 4 x 6 or 51 = 3 x 17 are examples." (Dunham 1994: 2)

trivial operation that is characterized by functional stagnation: it does not produce a quotient that is different from the dividend and the divisor: 1:1=1=1:1:1:…. Division of *two, three* and the other primes, on the other hand, is non-trivial: dividing a set with two elements into two results in two singletons: 2:2=1 and that is where the divisibility-by-two sequence stops. For the time being, we shall simply adopt the assumption that *one* is not a prime number, but we shall return to the issue below.

With this in mind, let us expand the set of numbers considered and add a second row:

(2)

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48

It was observed above that for all primes > 3 in the first row, primes occur on either side of the number 6 or a multiple of 6: they are *6n-adjacent*, with n ≥ 1. If we were to interpret this as meaning that all numbers on either side of the number 6 or a multiple of 6 are primes, the second row indicates that such an hypothesis is deficient. Both 25 and 35, which fit the description, are not primes, since they are both the products of two smaller whole numbers: 5 x 5 and 5 x 7, respectively. What this implies, is that some of the 6n-adjacent "possible" primes will have to be sieved out from the final set of actual primes.

In the table below, the set of numbers considered is expanded to the first 12 x 24 natural numbers, including:

(i) 6 and multiples of 6 (green underlined)

(ii) prime numbers (blue underlined)[2],

(iii) 'unexpected' non-primes, i.e. numbers one might have expected to be primes since they occur immediately before or after 6 or multiples of 6, but are not primes (red).

(3)

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96
	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
	121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143	144
	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160	161	162	163	164	165	166	167	168
	169	170	171	172	173	174	175	176	177	178	179	180	181	182	183	184	185	186	187	188	189	190	191	192
	193	194	195	196	197	198	199	200	201	202	203	204	205	206	207	208	209	210	211	212	213	214	215	216
	217	218	219	220	221	222	223	224	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239	240
	241	242	243	244	245	246	247	248	249	250	251	252	253	254	255	256	257	258	259	260	261	262	263	264
	265	266	267	268	269	270	271	272	273	274	275	276	277	278	279	280	281	282	283	284	285	286	287	288

[2] The first 62 prime numbers, i.e. if 1 is counted in, which we argued above plays a pivotal role in the system and fits the definition (1 is trivially divisible both by itself and by 1).

What is striking here, is

(a) that *possible primes* (= the blue underlined + the red numbers) are still always numbers just before and after 6 and multiples of 6;

(b) that the only prime numbers not fitting that description are 2 and 3;

(c) that 2 and 3 are precisely the numbers whose multiplication yields 6: 1 x 2 x 3 = 6;

(d) that the possible primes which ultimately turn out not to be primes are multiples of possible prime numbers.

It seems then

(a) that an algorithm for prime numbers involves three *primitives*, namely 1, 2 and 3, with 1 the basic additive number (natural number) and 2 and 3 the most basic multiplicative numbers (primes);

(b) that all the *possible prime numbers* are multiplications on the basis of 2 and 3, minus or plus pivot 1;

(c) that the red possible primes, which are to be kept out of the final set of *actual primes,* are multiplications involving possible prime numbers.

So let's have a stab at an algorithm.

(4) ***Minimalist algorithm for prime numbers***:

1. ***Primitives***: 1 (additive), 2 and 3 (multiplicative)
2. Determination of primes by (a) selection of the *possible primes* (b) *sieving out* from the set obtained those numbers which are multiplications of possible primes, which leaves the set of *actual primes.*
   a. ***possible prime SELECTION***: for $n \geq 1$:
      a) (2n x 3) - 1; or
      b) (2n x 3) + 1

   υ set of **possible primes**

   b. ***actual prime SIEVE***[3]: multiplications of possible primes are *sieved out* and thereby do not qualify as actual primes, but end up as non-primes: for $n, p \geq 1$ & $p \geq n$[4]

[3] Name chosen with a wink to "that interesting dilettante in matters mathematical" (SMITH, 1925, p. 5), the versatile scholar and librarian of the famous library of Alexandria, *Eratosthenes of Cyrene* (appr. 275-195 BC). Not only was he the first who estimated accurately the diameter of the earth, he also "worked on a method of finding prime numbers by sifting out the composite numbers in the natural series, leaving only primes. This he did by canceling the even numbers except 2, every third odd number after 3, every fifth odd number after 5, and so on, the result being what the ancient writers called the sieve." (SMITH, 1925, p. 5). Though different from our sieve in its effects, the idea that certain possible candidates for primehood have to be sifted out is very similar.

[4] The requirement $p \geq n$ is inserted to avoid that after sieving e.g. 5 x 7, a later operation will occur to sieve out 7 x 5.

sieve out ((2n x 3) +/- 1) x ((2p x 3p) +/- 1)
υ leaves the set of **actual primes**

This yields the set of primes.[5, 6]

Some examples to illustrate how this works:

2.a. possible primes: ((2x1) x 3) – 1 = 5; ((2x1) x 3) + 1 = 7; ((2x2) x 3) – 1= 11; ((2x2) x 3) + 1= 13, etc.

2.b. possible primes to be removed from the final set of primes: (((2x1) x 3) - 1) x (((2x1) x 3) - 1)= 5 x 5 = 25; (((2x1) x 3) - 1) x (((2x1) x 3) + 1)= 5 x 7 = 35; (((2x1) x 3) + 1) x (((2x1) x 3) + 1) = 7 x 7 = 49; (((2x1) x 3) - 1) x (((2x2) x 3) - 1) = 5 x 11 = 55, etc.

2.a. yields all numbers just before and after 6 and multiples of 6 (= the blue underlined numbers other than 1, 2 and 3 + the red numbers);

2.b. yields (in red in the table) all possible primes sieved out from the set of possible primes, which leaves the set of actual prime numbers generated by the algorithm, alongside the primitive primes 1 (pivot), 2 and 3.

The new features of the above analysis are:

1. A simple determination of the *primitive primes* in terms of the *most basic additive number 1 and the most basic multiplicative numbers 2 and 3*;
2. The definition of an algorithm which automatically generates all non-primitive primes (i.e. the primes > 3), an easy procedure making all calculation and laborious checking of individual numbers superfluous;
3. The two step nature (*selection* and *sieve*) of the determination of *actual primes;*
4. The notion *possible prime*, which makes it possible to distinguish preparatory possible prime selection from ultimate determination of the *actual primes;*
5. The actual prime *sieve*, which sifts out the non-primes from the set of possible primes, leaving the set of *actual primes.*

[5] To include 1 among the output of the rules for the determination of possible twin primes, one could choose to let n ≥ 0, so that ((2x0) x 3) + 1= 1. However, that cannot be correct for three reasons: (a) 1 is is independently needed as a primitive in the possible-prime-formulas; and (b) application of possible-prime-formula 2. a. a) would yield ((2x0) x 3) - 1= -1, which exceeds the lower boundary 0 of the set of natural numbers; (c) ((2x1) x 3) - 1) x ((2x0) x 3) + 1)= 5, which would entail that 5 would have to be removed from the list of possible primes by rule 2.b.

[6] To use all the primitives in the first part of the formula, one can resort to (1 x 2n x 3) – 1 and (1 x 2n x 3) + 1, but that does not change anything and since 1 is used in the second part anyway, this may (cf. Ockham's razor) be an unnecessary complication.

3 Proofs

Property: Every prime number > 3 can be written as 6k+1 or 6k-1
Proof:

- 6k is divisible by 6
- 6k+2 is even
- 6k+3 is divisible by 3
- 6k+4 is even
- Hence these forms are not prime numbers
- (PS: 6k+5 is of the same form as 6k-1)

Convention: a *pp* is a possible prime and is of the form 6k$\pm$1

Property: The product of two pp's is a pp
Proof:

- A pp is not divisible by 2, 3 or 6. Consequently, the product of two pp's does not have 2, 3 or 6 as divisors and consequently cannot be written as 6k, 6k+2, 6k+3, 6k+4.

Property: a pp only has pp-divisors
Proof:

- A pp is not divisible by 2, 3 or 6. Consequently, the factors of a factorisation of a pp do not have 2, 3 or 6 as divisors and hence cannot be written as 6k, 6k+2, 6k+3, 6k+4. They are pp's.

Consequence: a *non-prime* pp can be written as the product of two pp's.

4 Computer Implementation of the Sieve in PASCAL[7]

PRIEM_DJ.PAS

```
program priem;
  uses crt;

const N = 3000;
var pos_prime: array [0..32000] of boolean;
    k: longint;

procedure results;
```

[7] Thanks to Dirk Ghysels for crucial help with this section.

```
    var k: longint;
    begin
        writeln  ('prime numbers between 1 and ',N);
        writeln  ('-----------------------------------------------------------');
        for k: =1 to n do
        if pos_prime [k] then
            begin
                write (k, '   ');
            end;
        writeln;
    end;

procedure sieve_dj;
    var k,l, max :  longint;
    begin
        for k: =1 to n do
            pos_prime [k]: = false;
        pos_prime [1]: = true;
        pos_prime [2]: = true;
        pos_prime [3]: = true;
        l := n div 6;
        for k: =1 to l do
            begin
                pos_prime [6*k-1] := true;
                pos_prime [6*k+ 1] := true;
            end;
        max := (n+1) div 6;
        for k:= 1 to max do
            for l:= k to max div k do
                begin
                    pos_prime [(6*k-1)*(6l-1)] := false;
                    pos_prime [(6*k+1)*(6l-1)] := false;
                    pos_prime [(6*k-1)*(6l+1)] := false;
                    pos_prime [(6*k+1)*(6l+1)] := false;
                end
    end;

begin

    clrscr;
    writeln ('calculating prime numbers');
    writeln ('------------------------------------');
```

```
   writeln ('method Dany Jaspers');
   writeln ('----------------------------');
   sieve_dj;
   results;
   repeat until keypressed;
end.
```

5 The Status of 0 and 1

The number *zero* has no role to play in the prime number system. This is often the case in multiplicative systems, due to the fact that *zero* literally has an *annihilating* effect in such an environment: whatever is multiplied by zero is reduced to it. This fact has linguistic relevance. In Pietroski (2004) a convincing case is made for what he calls "conjunctivism", which says that *absolutely all* semantically relevant syntactic concatenation expresses conjunction. If indeed linguistic phrase markers (tree structures) are conjunctive – or equivalently in algebraic terms: multiplicative – operations at every Merge step, we have a direct explanation for the fact that Merge can never be a trivial operation involving a semantically null constituent, since that would destroy all previously introduced information, given that the intersection of the null set with any other set is the null set, the set-theoretic equivalent of the algebraic observation that multiplications involving a zero always yield zero itself[8].

In additive systems, on the other hand, the informativeness status of *zero* amounts to *stagnation*. It may sound surprising to mathematicians who cherish commutativity, but although *zero* is perfectly fine as a point of departure ($0 + 1 = 1$) in common sense addition contexts where there is a contextual change from the nonpresence of an entity to its presence, it is an uneconomical *addend* in everyday contexts when "added" to another number which functions as the *augend*, as in ($1 + 0 = 1$). From the viewpoint of natural cognition and natural language, it does not really add anything there, the sum being no different from the augend before the additive operation started[9].

It is important to realize that the above observation does not invalidate commutativity in scientific mathematics, nor the use of 0 as second element in an addition operation: the "nonnaturalness" of zero in particular contexts

[8] Note that this I am obviously not claiming that all multiplicative systems bar zero, only those subject to a constraint barring annihilation of information do. This is precisely the dividing line between natural multiplication – which is subject to such an economy constraint – and nonnatural, higher mathematics, where multiplication with zero is a possible option.

[9] As said, the context in which it is fine, is when it is the initial element of an addition. A good example of a system subject to informativeness is the Fibonacci-sequence 0, 1, 1, 2, 3, 5, 8, 13, etc., an additive number system of which 0 is the point of departure, but never the added quantity. Below, we shall further elaborate the notion "nonnaturalness" in the context of natural language and propose a constraint against nonnatural operations in systems subject to informativeness. Needless to say, such operations are perfectly fine in nonnatural (= scientific) mathematics, where no informativeness-at-every-step constraint applies.

is restricted to those additive systems which are subject to a condition requiring newly added material to increase informativeness. The claim defended here is that such a constraint is part of our most natural, common sense use of number expressions in our daily interactions with one another, i.e. in what one might justifiably call *natural mathematics,* which Devlin (2000) distinguishes from formal mathematics: the former he calls "formalized common sense", while the latter is characterized by formal definitions which are free to go against common sense if they do the formal mathematical job they were invented and hired for. Note that the difference pointed out here is reflected in the fact that in maths the notion of *natural number* is either taken to mean an element of the set {1, 2, 3, ...} – that is, the counting numbers or the positive integers – or an element of the set {0, 1, 2, 3, ...}, the non-negative integers. Typically, the former conception is used in elementary number theory (and natural counting processes), whil e the latter is preferred in the realms of mathematical logic, formal set theory, and computer science. While number theory includes lower numbers and counting, the other realms are not equally directly linked to such basic, common sense mathematical operations.

How about the status of the number *one*? Here again we have reason to believe that it has no role to play in the prime number system. Actually, this is fully in line with the most fundamental theorem of arithmetic:

(5) *Fundamental Theorem Of Arithmetic*: Any positive integer (**other than 1**) can be written as the product of prime numbers in one and only one way. (DUNHAM 1994, p. 3).

To understand the rationale for this theorem, let us have a careful look at the reason why *one* is normally excluded from the set of primes in formal mathematical circles and then add our own natural mathematical reason why it has to be barred. The main reason in formal mathematics is to do with the desire for unique factorization, i.e. unique decomposition of numbers into their prime factors. As stated above, "primes are the multiplicative building blocks from which all whole numbers are assembled" (DUNHAM 1994: 3).

"Primes play a role analogous to that of the chemical elements, for just as any natural compound can be broken into a combination of the 92 natural elements on the periodic chart (or the 100+ elements including those created in the laboratory), so, too, can any whole number be decomposed into its prime factors. A molecule of the compound we call water, H_2O, can be separated into two atoms of the element hydrogen and one atom of the element oxygen. Similarly, the compound (i.e. composite) number 45 can be broken into a product of two factors of the prime 3 and one of the prime 5. Mimicking water's chemical notation, we could write $45 = 3_2 5$, although mathematicians prefer the exponential form $45 = 3^2 \times 5$.

But arithmetic's fundamental theorem provides more than just a decomposition into primes. Equally critical is its guarantee of the uniqueness of such decompositions. If someone determines the prime factorization of 92,365 to be 5 x 7 x 7 x 13 x 29, then a colleague – working across the room or across the country, working today or a thousand centuries from now – *must* come up with precisely the same decomposition." (DUNHAM 1994, p. 4).

And this is where the problem with 1 comes in: "For if 1 were categorized as a prime, then the number 14, for instance, would have prime decomposition 14 = 2 x 7 as well as the *different* prime decompositions 14 = 1 x 2 x 7 and 14 = 1 x 1 x 1 x 2 x 7. The uniqueness of prime decomposition would vanish." (DUNHAM 1994, p. 4).

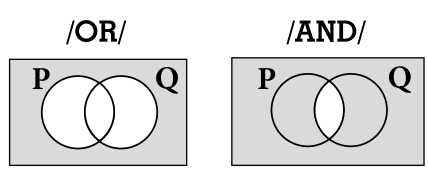
To see how *one* can be ruled out as a prime from a natural mathematical perspective too, a comparison with extra-mathematical natural language is instructive. In Jaspers (2005) we took inspiration from George Boole's (1854) work and established the following patterns of relations and modes of representation (linguistic, logical, set-theoretical and algebraic) for the sentences P: *John is in the garden* and Q: *Mary is in the garden*

(6)

	Language	**logic**	**set-theory**	**algebra**
/OR/	John is in the garden or Mary is in the garden	disjunction	union	addition
		$P \vee Q$	/P/[10] ∪ /Q/	P+Q
/AND/	John is in the garden and Mary is in the garden	conjunction	intersection	multiplication
		$P \wedge Q$	/P/ ∩ /Q/	PxQ

Venn-diagrammatic representations were provided to clarify the set-theoretic relationship between /OR/ and /AND/:

(7)



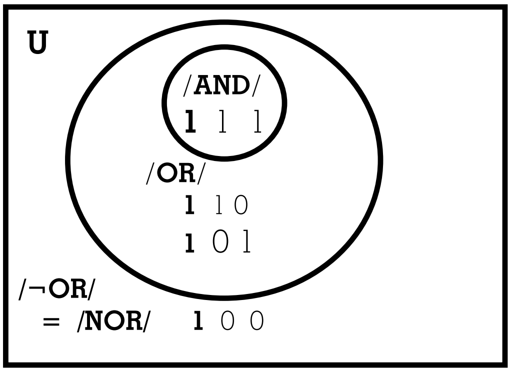
These representations illustrate clearly that from an extensional perspective, the set of situations where $P \wedge Q$ is true – the intersection of P and Q, when both P and Q are true – is a subset of the set of situations where $P \vee Q$ is true – the union of P and Q, i.e. when either P or Q or both are true[11], which can be summed up in the following valuation space diagram.[12]

[10] The extension of P is the set of possible situation in which P is true. This is called the *valuation space* of P, represented as /P/ (VAN FRAASSEN 1971, SEUREN 1998: 331, SEUREN et. al. 2001).

[11] For discussion of the distinction between inclusive /OR/ (represented here) and exclusive /OR/, Cf. chapter 1.

[12] We used Polish notation, so that the truth value in bold is that of the complex proposition, whereas the other two are the truth values of the simple propositions P and Q. The label U represents the Universe of possible situations.

(8)



Discourse, now, is conjunctive (or, to state it in algebraic terms: multiplicative), as is evident from the following pair of examples:

(9) John was in the garden (= A). Mary came in (= B).
John was in the garden *and* Mary came in.

The set of possible situations in which A∧B is true is a proper subset of the set of possible situations in which A alone is true: /A/∩ /B/ is a proper subset of /A/.

Now compare the previous pair with the next one:

(10) ??John was in the garden. John was in the garden.
??John was in the garden *and* John was in the garden.

This pair illustrates that in multiplicative systems like discourse, there is a requirement at work which demands what might be called *information increase*. Each new sentence should add information:

(11) ***Information Increase Requirement***:
An utterance P uttered in discourse context C must be informative in C

A first approximation of the notion informativeness runs as follows:

(12) ***Informativeness***
An utterance P is informative in the context C iff
CP, i.e. /C/∩ /P/ is a proper subset of /C/ and /C/ ∩ /P/ ≠ the empty set ø[13]

[13] The generalizations captured in (11) and (12) are based on Scharten (1997: 64). Yet, in view of Boolean insights, (12) is crucially reformulated in terms of conjunction/multiplication (rather than Scharten's disjunction/addition). Further on, a more radical reformulation of these notions will be proposed.

As indicated, the progression in discourse from A to B in (16) results in an implicit conjunction of their meanings at the point when B is introduced.

(13)

		conjunction	intersection	multiplication
"/AND/"[14]	John was in the garden > John was in the garden. (*And*) Mary came in.	$A > A \wedge B$	$/A/ > /A/ \cap /B/$	$A > AxB$

As was explained in Jaspers (2005) and is briefly repeated here, this approach provides a solution to a problem with logical entailment and so-called nonnatural entailments.

(14)

P: *All* flags are green
Q: *Some* flags are green

Of this pair of sentences, the first logically entails the second (or, in formal notation P |– Q): whenever P is true, Q must of necessity also be true, on account of the meanings of the logical constants *all* and *some* involved in P and Q. The set-theoretic definition in terms of set-inclusion runs as follows:

(15) Logical entailment

For all sentences X and Y, X |– Y iff $/X/ \subseteq /Y/$

However, this definition turned out to be insufficiently restrictive from the viewpoint of natural language: it posits entailment-relations between certain sentences for which it strains linguistic intuition to claim that the one sentence "follows" logically from the other.

(16)

a. Ex necessarie falso sequitur quodlibet	C :Some rhinoceros is not a rhinoceros |– P: Some senile professor has pink stockings
b. Verum sequitur ad quodlibet	C: John is in the garden |– P: A rose is a rose

(16) a. illustrates that a contradiction entails everything. Since the contradiction is never true, the logical entailment relation holds trivially. In (16) b., it does not matter to the entailment relationship whether the entailer-sentence is true or not. The entailed sentence being a necessary truth, i.e. a sentence that is true in all possible situations, the status of the entailer-sentence cannot affect the entailment-relation, which obtains in any case.

[14] The quotation marks and indicate the implicit nature of conjuntion (at last when the word and is not explicity used).

In *ex necessarie falso sequitur quodlibet*-cases the proposition that is responsible for the nonnaturalness is the contradiction - whose extension is the null set Ø –. As it is true in no situation at all, its extension is the empty valuation space. In *verum sequitur ad quodlibet*-cases, the troublesome proposition is the necessary truth, whose extension is a value at the other extreme: the total valuation space U, the entire universe of possible situations. By appeal to the Information Increase Requirement – the requirement that meaning has to be added at every step – these nonnatural entailments out are easily ruled out.

(17)

Ex necessarie falso sequitur quodlibet	/C/ = Ø	Ø ∩ /P/ = Ø, hence P is not informative in /C/
Verum sequitur ad quodlibet	/P/ = U	/C/ ∩ U = /C/, hence P is not informative in /C/

Actually, these observations can be generalized: Ø and U have no incremental or informative potential in *any* conjunctive-multiplicative context subject to informativeness. "Nonnatural entailments" in natural language caused by contradictions and universal truths are therefore but one illustration of nonnaturalness in multiplicative systems subject to informativeness.

(18)

Non-informativeness in any context For *any* /C/:	**Set-theory**	**Algebra**	**Type of non-informativeness**
	*/C/ ∩ Ø = Ø	*C x 0 = 0	Information annihilation
	*/C/ ∩ U = /C/	*C x 1 = C	Information stagnation

In view of the above, (11) and (12) can now be collapsed into a single *economy principle* barring non-informative operations in natural multiplicative systems. This principle has the added benefit over (11) and (12) of being expressed in negative terms, i.e. it comes out as a filtering device or constraint, rather than as a positive normative injunction. Such a negative statement fares better, being a more direct expression of the bounds of multiplicative systems subject to informativeness than (11) and (12): it cuts out what is nonnatural, leaving the rest unaffected.

(19)

The Non-informative Operations Constraint (NOC) For *any* /C/, /P/:	**Set-theory**	**Algebra**	**Type of non-informativeness**
	*/C/ ∩ /P/ = Ø	*C x P = 0	Information annihilation
	*/C/ ∩ /P/ = /C/	*C x P = C	Information stagnation

Equipped with the algebraic formulations of (18) and (19) we can return to mathematics, since if we appeal to NOC for prime factorization, we get precisely the desired results. First of all, zero cannot be used informatively in factorizations (and in NOC-sensitive multiplicative systems in general), since it annihilates all information. Moreover, the number 1 cannot be used in factorizations either, because it leads to information stagnation.[15] It is ruled out in natural mathematics by the same economy considerations that rule out (10) as "nonnatural". Note that nonnaturalness does not imply that we cannot consciously choose to violate the principle for the purpose of naturalistic mathematical inquiry in formal mathematics. Actually, much the same is also true of grammatical principles in natural language and natural logic: we can construct nonnatural entailments fully satisfying the definition of entailment – we have done so above – just as much as we can multiply by 1 (and by 0 for that matter) if we choose to. Neither operation is incoherent. Actually, in the Boolean (1854) algebra of 0 and 1, which is not subject to NOC, it is even crucial. For factorization, however, one has to bear in mind that it leads to lack of informativeness and nonnaturalness, which is why it is ruled out in the multiplicative part of the natural prime algorithm, ultimately for reasons of economy ("do not perform any operation which does not increase informativeness"). And to conclude the argumentation: just as nonnatural entailments do not lead to the conclusion that the propositions contained in them are no longer propositions, nonnatural factorizations such as $14 = 1 \times 2 \times 7$ and $14 = 1 \times 1 \times 1 \times 2 \times 7$ do not warrant the conclusion that 1 could not for the sake of formal mathematical exploration be proposed to be a (trivial) prime. Actually, there have been proposals along those lines in the history of mathematics. Our arguments in favour of a dividing line between natural and nonnatural realms in mathematics, natural language and logic alike render further support to the independently established (Boolean) insight that mathematics, language and logic are not disjoint. If we are right, the natural variant of each is in an important respect governed by the same laws of an internalist, cognitive nature, entirely in the spirit of Boole's Laws of Thought (1854).

6 Mathematics, Logic and Natural Language

Concerning the relationship between mathematics, logic and natural language, it is worth mentioning that the above multiplicative prime algorithm is predicated on and hence presupposes the additive system of natural numbers. Both possible prime selection and the actual prime sieve need $n, p \geq 1$, i.e. the sequence of natural numbers (except zero), as their input. This confirms Dunham's (1994: 3) view that in the prime number system, additive creations are examined under a multiplicative light. But more important to our analysis, it also reinforces

[15] It is for the same reasons that 0 has no role to play in the formulation of our prime algorithm. In the multiplicative part of the formula, it would have an annihilating effect, in the additive/subtractive part it would have a stagnation effect. While 1 cannot play a role in the multiplicative part of the formula, it does play a role as a building block in the additive (and its inverse subtractive) part. NOC explains why additive 1 and multiplicative 2 and 3 are useful elements in the algorithm, while 0 is useless and 1 confined to the additive/subtractive part. Any other constellation would cause non-informativeness.

certain conclusions arrived at in Jaspers (2005). We there posited that the conjunction *and* results from adding a meaning specification to the meaning of *or* and pointed to Boole's view that logical conjunction is algebraically a case of multiplication, while disjunction is a case of addition. It is clear, now, that the observation in this section that the multiplicative prime system is predicated on the additive natural number system is in perfect unison with those claims regarding the relation between the two propositional operators and their natural language equivalents, the lexical items *or* and *and*. And there is a further parallel. The analysis above has brought to light that natural numbers and primes are intertranslatable: even though the conjunctive-multiplicative system of primes is predicated on a disjunctive-additive system (the natural numbers) by the minimalist algorithm and hence, the latter are more basic than the former, this does not prevent factorization of natural numbers into products of primes. Analogously, though disjunctive *or* was proven in Jaspers (2005) to be more basic, both in language and logic, than conjunctive *and*, these two operators are also well-known to be intertranslatable: $P \wedge Q$ is equivalent to $\neg(\neg P \vee \neg Q)$, and $P \vee Q$ to $\neg(\neg P \wedge \neg Q)$. These are interesting, but no longer surprising conclusions once it is assumed that multiplicative and additive number systems subject to NOC, conjunction and disjunction in natural language and the system of propositional operators are all emanations of the same underlying system.[16]

7 Computational Ease and Algorithmic Simplicity

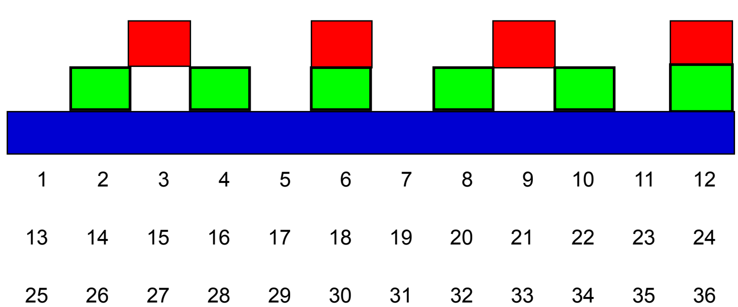
To formulate *the possible prime selection* rule and to implement it computationally, it may be marginally simpler to use the formula (6 x n) +/– 1, as in the computer implementation above, rather than (2n x 3) +/– 1. However, our interest is in as minimal a system as possible, generating everything from the *three primitives one* (additive), *two* and *three* (multiplicative) and also and crucially to *stay within the paradigm of prime numbers in the multiplicative part*, to which 6 does not belong. Our first concern is cognitive reality and (given that there is mounting evidence from linguistic research that linguistic rules are very minimal) optimal simplicity of the algorithm. Though it is very interesting to see that the algorithm can be turned into a little computer programme on the basis of (6 x n) +/– 1, our concern for ease of computation is secondary to the empirical question which formulation stands more chances of being cognitively real. In other words, the hypothesis as formulated amounts to the claim that the mind operates with no more than three primitives and a selection rule and sieve formulated in terms of them.

Now, are there indications that the primitives used are indeed innate concepts? For the distinction between 1 and > 1 there is of course ample independent linguistic evidence universally in view of the linguistic distinction between singular and plural. No natural language does without that distinction, which at

[16] According to Jaspers (2005), this is an integrated system of 'molecular' operators built on the basis of a single negative operator NEC, better known as Peirce's dagger. For ample discussion, see chapters 1 and 2.

least makes this distinction a candidate for being part of universal grammar (UG) in the Chomskyan sense. If our proposal is correct, the distinction between the numerosities 2 and 3 also has to be claimed to be part of our basic mental mathematical toolkit. That this might be the case is independently suggested by numerosity research: "Man, even in the lower stages of development, possesses a faculty which, for want of a better name, I shall call *number sense*. This faculty permits him to recognize that something has changed in a small collection when, without his direct knowledge, an object has been removed or added to a collection." (Dantzig 1954, cited in Devlin 2000, 21). The restriction to a "small collection" is crucial: as argued by Butterworth (i.a.), "the Number Module is the innate core of our numerical abilities – a numerical 'start-up kit'. It categorizes the world in terms of numerosities, up to about 4 or 5" (Butterworth 1999, 8). That is even more than what we need for the algorithm. Note that there exist so-called one-two-many languages, in which any cardinality above two is given the same vague name. This means that such languages distinguish three basic types of cardinalities: one, two, at least three. Many more languages are one, two, three, (sometimes four, five), many languages, in which case those numbers that exist above three are approximative rather than precise. All of this supports the idea that there is indeed a set of primitives that functions as a starting kit, a minimal distinction between three cardinalities. The distinction between one, two and many seems to be universal; three is also very widespread but not universal and from there on you have what your culture cared or needed to digitalize beyond that. Note another interesting parallel in a related quantificational linguistic domain: all languages have an equative and a comparative degree of comparison, many but not all a separate superlative.

That the whole system of possible primes needs no more numerical information than the status of divisibility by 2 (green) and 3 (red) is illustrated in the following representation. Divisibility by 1 (blue) is irrelevant since all numbers can be divided by 1. The crucial issue is divisibility by 2 and by 3. It is those numbers which are neither divisible by 2 nor by 3 that are possible primes: 5 and 7, 11 and 13, and so on: numbers adjacent to 6 and multiples of 6. In brief, two few very low numbers – arguably natural and innate - are all that is needed to yield the whole set of possible primes.



On the whole, algorithm (4) does not throw a new light on unsolved problems and open questions relating to (high) prime numbers, nor do I think it opens a route for new theorems about the complexities of prime numbers. But that is not an interesting issue in natural mathematics, whose key concern is lower maths. Logically speaking, a minimalist algorithm for human beings' prime number *competence* need not have any implications nor represent progress in the area of *performance* problems[17] incurred when very large (hence non-common-sense) numbers have to be dealt with, numbers beyond 10,000,000,000, for instance (Ignace Vandewoestyne (p.c.). In those large number domains, two types of problems arise, depending on the nature of the algorithm employed: either there are time problems – i.e. there are so many calculations to do that even fast supercomputers cannot carry them out within a reasonable time frame – or there are memory problems, when so many interim calculations have to be committed to memory that it leads to overload. Given that our sieve has to expurge all products of possible primes, it is quite possible that the type of problem that will bedevil it in the area of large numbers will be the time problem. Yet the problems should not be as serious as those faced by Eratosthenes' sieve. The latter is actually a collection of several sieves: to find all primes less than 100, for instance, it needs 4 sieves. Since every composite integer less than 100 must have a prime factor less than the square root of 100 (= 10), we need to check each integer less than 100 for divisibility by primes (except 1) less than 10, i.e. by 2, 3, 5, 7 and consequently sieve out all multiples of 2, 3, 5 and 7. This is of course extremely inefficient and laborious, in a way the minimalist algorithm is not. The reason is that the possible prime selection rule is stated *positively*, i.e. it selects all candidates for primehood in one rule. That all the other numbers are excluded follows from their non-selection by the possible-prime rule, but does not have to be checked independently. Only the actual-prime sieve requires a lot of calculation and might eventually run into trouble when extremely high numbers have to be dealt with. We shall gladly leave this matter to specialists in mathematics and computer science to grapple with. To their taste, the present algorithm may not represent a decisive improvement over other existing algorithms, but to a linguist it might. Linguists are less interested in the nature of the intricacies that make, say, Joyce's *Ulysses* different from everyday language use than in the most plausible underlying mental system that sets both Joyce and everyday language users off from beings that lack the natural language (and number) capacity.

Popper's question whether or not the twin primes fizzle out will consequently also be left unanswered. Though the set of natural numbers is infinite and (given the algorithm) the set of possible primes is infinite too, the same is true for the set of removed possible primes, so I cannot judge what the outcome will be for lack of

[17] *Performance* as used here is a technical term, not the common sense term with its erroneous connotation of effortlessness ("that's merely a performance problem, an execution problem"). Anybody who makes efforts to develop an innate musical competence into advanced musical skills (learning how to play the piano or the violin well for instance), knows full well how much effort, dedication and repetitive practice is required to turn an innate competence into advanced performance skill. Within the realm of performance, there is a further distinction between proficient imitatio (driven by aemulatio) and original inventio. The latter invokes creativity that may well be beyond human powers of description, a feature it shares with other cognitive powers such as (i.a.) the science-forming capacity.

knowledge of maths. Judging from the small set of numbers in the table above, the number per 24 stays pretty constant: while in the first row there are three twin primes, from the second row there are two or more commonly one[18]. But of course this is far too small a set to warrant any serious conclusions.

(20)

# twin																								
3	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
1	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
1	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96
2	97	98	99	100	101	102	103	104	105	106	107	108	109	110	111	112	113	114	115	116	117	118	119	120
1	121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143	144
1	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160	161	162	163	164	165	166	167	168
1	169	170	171	172	173	174	175	176	177	178	179	180	181	182	183	184	185	186	187	188	189	190	191	192
2	193	194	195	196	197	198	199	200	201	202	203	204	205	206	207	208	209	210	211	212	213	214	215	216
1	217	218	219	220	221	222	223	224	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239	240
1	241	242	243	244	245	246	247	248	249	250	251	252	253	254	255	256	257	258	259	260	261	262	263	264
2	265	266	267	268	269	270	271	272	273	274	275	276	277	278	279	280	281	282	283	284	285	286	287	288

A somewhat larger selection (36 x 36) is represented in the appendix, with the number of twin primes in the leftmost column. Even if it does not help resolve the twin prime problem, the selection still provides a grid for those who wish to check the algorithm proposed above against the restricted set of numbers collected here. The more art-minded readers are invited to look at it as a lively splash of colours in a number pattern. Surely it cannot match Jasper John's *The Number Zero*, but it is prime minimalism to me.

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[18] When a twin prime occurred on the transition from one row to the next, it was counted as part of the latter, since its twin prime status could only be determined at that point.

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Appendix: 36 x 36 natural numbers, including actual primes, sifted out possible primes and number of twin primes per row

Blue: actual primes
Red: sieved out possible primes
First column: number of twin primes

# twin primes																																				
4	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
2	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
2	73	74	75	76	77	78	79	80	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100	101	102	103	104	105	106	107	108
2	109	110	111	112	113	114	115	116	117	118	119	120	121	122	123	124	125	126	127	128	129	130	131	132	133	134	135	136	137	138	139	140	141	142	143	144
1	145	146	147	148	149	150	151	152	153	154	155	156	157	158	159	160	161	162	163	164	165	166	167	168	169	170	171	172	173	174	175	176	177	178	179	180
3	181	182	183	184	185	186	187	188	189	190	191	192	193	194	195	196	197	198	199	200	201	202	203	204	205	206	207	208	209	210	211	212	213	214	215	216
2	217	218	219	220	221	222	223	224	225	226	227	228	229	230	231	232	233	234	235	236	237	238	239	240	241	242	243	244	245	246	247	248	249	250	251	252
2	253	254	255	256	257	258	259	260	261	262	263	264	265	266	267	268	269	270	271	272	273	274	275	276	277	278	279	280	281	282	283	284	285	286	287	288
1	289	290	291	292	293	294	295	296	297	298	299	300	301	302	303	304	305	306	307	308	309	310	311	312	313	314	315	316	317	318	319	320	321	322	323	324
1	325	326	327	328	329	330	331	332	333	334	335	336	337	338	339	340	341	342	343	344	345	346	347	348	349	350	351	352	353	354	355	356	357	358	359	360
0	361	362	363	364	365	366	367	368	369	370	371	372	373	374	375	376	377	378	379	380	381	382	383	384	385	386	387	388	389	390	391	392	393	394	395	396
1	397	398	399	400	401	402	403	404	405	406	407	408	409	410	411	412	413	414	415	416	417	418	419	420	421	422	423	424	425	426	427	428	429	430	431	432
2	433	434	435	436	437	438	439	440	441	442	443	444	445	446	447	448	449	450	451	452	453	454	455	456	457	458	459	460	461	462	463	464	465	466	467	468
0	469	470	471	472	473	474	475	476	477	478	479	480	481	482	483	484	485	486	487	488	489	490	491	492	493	494	495	496	497	498	499	500	501	502	503	504
1	505	506	507	508	509	510	511	512	513	514	515	516	517	518	519	520	521	522	523	524	525	526	527	528	529	530	531	532	533	534	535	536	537	538	539	540
1	541	542	543	544	545	546	547	548	549	550	551	552	553	554	555	556	557	558	559	560	561	562	563	564	565	566	567	568	569	570	571	572	573	574	575	576
1	577	578	579	580	581	582	583	584	585	586	587	588	589	590	591	592	593	594	595	596	597	598	599	600	601	602	603	604	605	606	607	608	609	610	611	612
2	613	614	615	616	617	618	619	620	621	622	623	624	625	626	627	628	629	630	631	632	633	634	635	636	637	638	639	640	641	642	643	644	645	646	647	648
1	649	650	651	652	653	654	655	656	657	658	659	660	661	662	663	664	665	666	667	668	669	670	671	672	673	674	675	676	677	678	679	680	681	682	683	684
0	685	686	687	688	689	690	691	692	693	694	695	696	697	698	699	700	701	702	703	704	705	706	707	708	709	710	711	712	713	714	715	716	717	718	719	720
0	721	722	723	724	725	726	727	728	729	730	731	732	733	734	735	736	737	738	739	740	741	742	743	744	745	746	747	748	749	750	751	752	753	754	755	756
0	757	758	759	760	761	762	763	764	765	766	767	768	769	770	771	772	773	774	775	776	777	778	779	780	781	782	783	784	785	786	787	788	789	790	791	792
2	793	794	795	796	797	798	799	800	801	802	803	804	805	806	807	808	809	810	811	812	813	814	815	816	817	818	819	820	821	822	823	824	825	826	827	828
2	829	830	831	832	833	834	835	836	837	838	839	840	841	842	843	844	845	846	847	848	849	850	851	852	853	854	855	856	857	858	859	860	861	862	863	864
1	865	866	867	868	869	870	871	872	873	874	875	876	877	878	879	880	881	882	883	884	885	886	887	888	889	890	891	892	893	894	895	896	897	898	899	900
0	901	902	903	904	905	906	907	908	909	910	911	912	913	914	915	916	917	918	919	920	921	922	923	924	925	926	927	928	929	930	931	932	933	934	935	936
0	937	938	939	940	941	942	943	944	945	946	947	948	949	950	951	952	953	954	955	956	957	958	959	960	961	962	963	964	965	966	967	968	969	970	971	972
0	973	974	975	976	977	978	979	980	981	982	983	984	985	986	987	988	989	990	991	992	993	994	995	996	997	998	999	1000	1001	1002	1003	1004	1005	1006	1007	1008
2	1009	1010	1011	1012	1013	1014	1015	1016	1017	1018	1019	1020	1021	1022	1023	1024	1025	1026	1027	1028	1029	1030	1031	1032	1033	1034	1035	1036	1037	1038	1039	1040	1041	1042	1043	1044
2	1045	1046	1047	1048	1049	1050	1051	1052	1053	1054	1055	1056	1057	1058	1059	1060	1061	1062	1063	1064	1065	1066	1067	1068	1069	1070	1071	1072	1073	1074	1075	1076	1077	1078	1079	1080
1	1081	1082	1083	1084	1085	1086	1087	1088	1089	1090	1091	1092	1093	1094	1095	1096	1097	1098	1099	1100	1101	1102	1103	1104	1105	1106	1107	1108	1109	1110	1111	1112	1113	1114	1115	1116
0	1117	1118	1119	1120	1121	1122	1123	1124	1125	1126	1127	1128	1129	1130	1131	1132	1133	1134	1135	1136	1137	1138	1139	1140	1141	1142	1143	1144	1145	1146	1147	1148	1149	1150	1151	1152
1	1153	1154	1155	1156	1157	1158	1159	1160	1161	1162	1163	1164	1165	1166	1167	1168	1169	1170	1171	1172	1173	1174	1175	1176	1177	1178	1179	1180	1181	1182	1183	1184	1185	1186	1187	1188
0	1189	1190	1191	1192	1193	1194	1195	1196	1197	1198	1199	1200	1201	1202	1203	1204	1205	1206	1207	1208	1209	1210	1211	1212	1213	1214	1215	1216	1217	1218	1219	1220	1221	1222	1223	1224
1	1225	1226	1227	1228	1229	1230	1231	1232	1233	1234	1235	1236	1237	1238	1239	1240	1241	1242	1243	1244	1245	1246	1247	1248	1249	1250	1251	1252	1253	1254	1255	1256	1257	1258	1259	1260
2	1261	1262	1263	1264	1265	1266	1267	1268	1269	1270	1271	1272	1273	1274	1275	1276	1277	1278	1279	1280	1281	1282	1283	1284	1285	1286	1287	1288	1289	1290	1291	1292	1293	1294	1295	1296