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The metalogical hexagon of opposition

Dedicated to my friends of Ceara

ABSTRACT

The difference between truth and logical truth is a fundamental distinction of modern logic promoted by Wittgenstein. We show here how this distinction leads to a metalogical triangle of contrariety which can be naturally extended into a metalogical hexagon of oppositions, representing in a direct and simple way the articulation of the six positions of a proposition vis-à-vis a theory. A particular case of this hexagon is a metalogical hexagon of propositions which can be interpreted in a modal way. We end by a semiotic hexagon emphasizing the value of true symbols, in particular the logic hexagon itself.

Keywords: Hexagon of opposition; triangle of contrarities; Logical truth; Wittgenstein; Philosophy of Logic.

RESUMO

A diferença entre verdade e verdade lógica é uma distinção fundamental promovida por Wittgenstein. Mostramos aqui como esta distinção leva a um triangulo metalógico da contrariedade que pode ser naturalmente estendido num hexágono metalógico de oposições, que representa de uma forma direta e simples as seis posições de uma proposição relativamente a uma teoria. Um caso particular deste hexágono é um hexágono metalógico de proposições que pode ser interpretado de uma forma modal. Terminamos com um hexágono semiótico apontando o valor dos verdadeiros símbolos, como o próprio hexágono lógico.

Palavras-chave: Hexágono de oposicão; triângulo de contrariedades; verdade lógica; Wittgenstein; Filosofia da Lógica.

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1 Truth and logical truth

One fundamental advance in modern logic was the distinction between *truth* and *logical truth*. This distinction was clearly expressed by Wittgenstein through the notion of *tautology* (*Tractatus* 4.46). A tautology is a proposition which is always true. Wittgenstein conceptualized this notion with the help of the notion of *truth-possibilities* (*Wahrheitsmöglichkeiten* in German). A truth-possibility is now called a bivaluation, or distribution of truth-values when restricted to atomic propositions.

Within this framework, beyond the duality truth/logical truth, we have a triangle articulating three situations:

(1) Propositions that are always true
(2) Propositions that are always false
(3) Propositions that can be true and can be false.

The first situation (1) has been characterized by Wittgenstein with the word "tautology". Wittgenstein did not invent this word, but he gave to it a meaning that is now strongly attached to it. But not necessarily to Wittgenstein: some people use this word in the sense of Wittgenstein without knowing that it is due to him. This is for example the case of Saunders MacLane in his book *Mathematics, form and function*, where he strongly criticizes Wittgenstein's philosophy of mathematics, arguing that Ludwig had a knowledge of mathematics reduced to high school mathematics, but at the same time promotes the notion of tautology and mathematical form as the fundamental aspect of mathematics.

The tandem Wittgenstein-Tautology can maybe compared to the tandem Heidegger-Dasein. Heidegger also did not create a new word, but he attached to "Dasein" a new fundamental meaning, so that it is difficult nowadays to speak about *Dasein* without thinking of Heidegger. On the other hand there is a difference between Wittgenstein and Heidegger in the sense that Wittgenstein gave a precise definition of what is a tautology, which has autonomy and was furtherdeveloped independently of him. This is probably why in the case of *tautology* what is attached to the word is the meaning given by Wittgenstein (or an evolution of it) rather than the name Wittgenstein. In the case of *Dasein* this is rather the name Heidegger since the meaning is far to be clear outside heideggerianism.

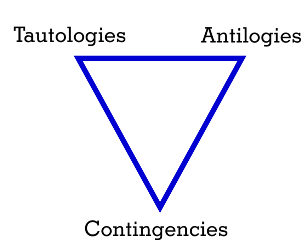
This is one of the important differences between science and philosophy. What is very interesting with Wittgenstein's notion of tautology is that it has a double aspect: mathematical and philosophical. The mathematical definition is that a proposition is a tautology if and only if it is true for all bivaluations, i.e. under any homomorphism between the algebra of the language and the algebra of truth-values, as it was later on precisely characterized in the case of truth-functional semantics (see Beziau 2012b for more details). But Wittgenstein also simultaneously gave a strong philosophical characterization of this notion related to the whole philosophical framework of truth-values and propositions. He claimed that a tautology says nothing about the world. He insisted that when we say "It is raining or it is not raining", we say nothing about the world (*Tractatus* 4.461).

Due to the absence of meaning of a tautology,[1] Wittgenstein was even led to claim that tautologies are not really propositions (*Sätzen*). And he said the same about propositions which are always false. The terminology for those propositions, the case (2), was not so happily chosen by Wittgenstein. In the *Tractatus* he called them "contradictions" (*Tractatus* 4.46). In fact in classical logic a contradiction is always false, but we can conceive propositions which are always false and are not contradictions and contradictions which are not always false. This is the distinction between triviality and contradiction, which can be considered as the essence of paraconsistent logic, as promoted by Newton da Costa (1958). In this second period Wittgenstein was in favour of this distinction and for this reason has been considered as a forerunner of paraconsistent logic.[2] A proposition that is always false can be called a triviality, but a better name, if we want to establish a connexion with tautology, is the word "antilogy".

For Wittgenstein only in the third situation (3), we have propositions which are real propositions because they have a meaning: they can be true or can be false, therefore say something about the world. Wittgenstein didn't give a specific name for such propositions, apart claiming that they are the only true propositions. Nowadays we can call these propositions, contingent propositions, or for, short, contingencies.

2 The metalogical triangle of contrariety

The three situations put forward by Wittgenstein can be described by the following triangle:

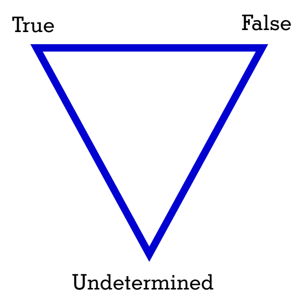


[1] Wittgenstein says "Tautologies and contradictions lack sense" (*Tractatus* 4.461) but also "Tautologies and contradictions are not, however, nonsensical" (*Tractatus* 4.4611).We will not emphasize here the distinction between « lack of sense » and « nonsensical » which may lack of sense or be nonsensical.

[2] See Wittgenstein, 1939, in particular his discussion with Turing on contradictions. Goldstein (1986) presents a very good analysis of the evolution of Wittgenstein's views on contradiction.

This triangle is a triangle of contrariety: the relation between any pair of vertices is a relation of contrariety. This means that a proposition cannot be at the same time a tautology and a contingency, a tautology and an antilogy, an antilogy and a contingency.

We can call this triangle a *metalogical triangle*, although for Wittgenstein himself this terminology would have been a monstrosity (see e.g. Padilla Gálvez, 2005). "Metaphysics" is a quite monstrous word having an ambiguous meaning and the recent proliferation of the prefix "meta" is also something quite absurd: "metaethics", "metaesthetics", "metaphilosophy", "metametaphyics" … However like with "metamathematics", the meaning of "metalogic" here is fairly clear: this is a theory articulating basic logical notions. In some sense we can say that this is logic, but to call this triangle, simply a *logic triangle* may lead to some misunderstandings. Someone may want to call this triangle, a *truth triangle*. This makes sense because this triangle is basically about truth. But this would create a confusion with the following triangle going beyond the dichotomy true/false:



This triangle is also a triangle of contrariety. It is connected with three-valued logic. Using it we can define a metalogical triangle similar to the one of classical logic, considering for example that a tautology is a proposition which is always designated (true), an antilogy a proposition that is always non-designated (false or undetermined) and a contingency a proposition that can be designated (true) or not designated (false or undetermined). This is the idea of Lukasiewicz's three-valued logic (1920) which has therefore the same basic metalogic as classical logic, i.e. the division of propositions in three classes. If we consider the paraconsistent interpretation of the above triangle, according to which undetermined is designated (Asenjo 1954), we still have these three classes: a tautology is a proposition that is always designated (true or undetermined), an antilogy a proposition that is always non-designated (false), a contingency a proposition that can be deginated (true or undetermined) or non-designated (false).

This metalogical structure can in fact be considered for many non-classical logics based on logical matrices, possible world semantics, or other tools. We can even consider this triangle as a fundamental structure not based on some primitive notions, keeping the intuitions of tautology, antilogy and contingency just defined abstractly with this triangle.

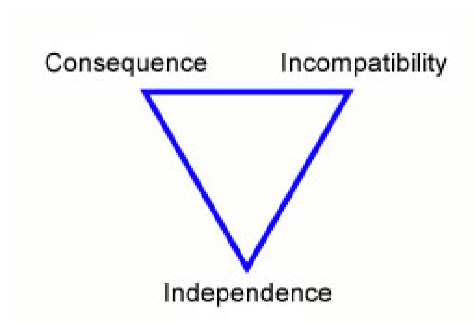
This triangle can be generalized to consequence relations. We have three different situations:

(1) A proposition is always a consequence of a theory
(2) A proposition is never a consequence of a theory
(3) A proposition can or not be a consequence of a theory

We can use the following terminology to describe these three situations of a proposition vis-à-vis a theory:

(1) Consequence
(2) Incompatibility
(3) Independence

We have then the following triangle.



In case of classical logic, we can use negation to describe these three situations with the symbol $\models$. Given a theory T, a proposition p, and the negation of this proposition $\neg p$, we have the following three situations:

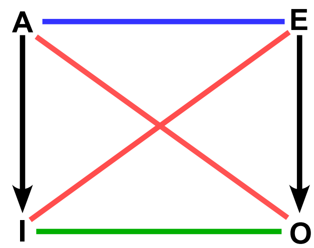
(1) $T \models p$
(2) $T \models \neg p$
(3) $T \not\models \neg p$ and $T \not\models p$

$T \not\models \neg p$ means that there is a model of T in which $\neg p$ is false, which means according to classical negation that p is true. According to this model, p is a consequence of T. On the other hand $T \not\models p$ means that there is a model of T according to which p is not a consequence of T.

These three situations do not describe the full picture, since we may have that a proposition is sometimes simply not a consequence of a theory. There is a model of the theory in which the proposition is false: the proposition is falsifiable, which can symbolically be written as $T \not\models p$. We don't need to use negation here and this situation makes sense independently of the existence of a classical negation, for any logic.

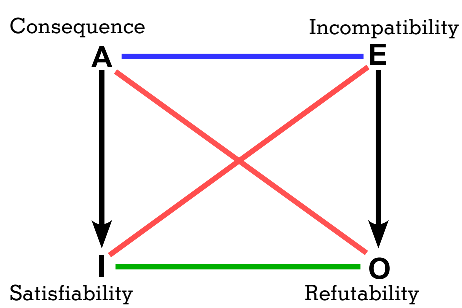
3 The metalogical square of opposition

A way to describe falsifiability is to use the square of opposition. Let us remember that the square of opposition is a theory that can be expressed by the following diagram:



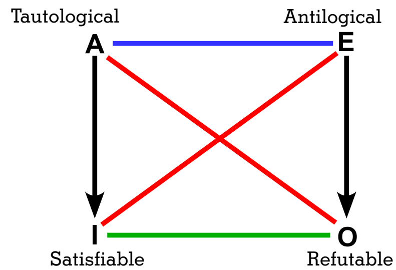
The four edges and the two diagonals of the square represent four relations between these propositions: red is the relation of *contradiction*, blue the relation of *contrariety*, red the relation of *subcontrariety*, black the relation of *subalternation*. These relations are defined as follows: two propositions are said to be *contradictory* iff they cannot be true and cannot be false together, *contrary* iff they can be false together but not true together, *subcontrary* iff they can be true together but not false together. A proposition is said to be *subalterned* to another one, if it is implied by it.

We can draw the following square for logical consequence, where the notion of refutability appears as well as the well-known notion of *satisfiability*:



According to this diagram, satisfiability and refutability are subcontaries, this means that *p is satisfiable* and *p is refutable* are two propositions that can be true at the same time, but cannot be false at the same time. This way of speaking involved propositions about propositions, this is one of the reasons to use the prefix "meta", calling this square a metalogical square. One could avoid in a way this situation defining a square of opposition directly on concepts rather than on propositions. This metalogical square of consequence describes four positions of a proposition vis-à-vis a theory and is the basis of consequence relation not only for classical logic but for a wide range of logics.

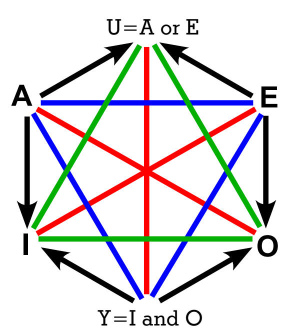
A particular case of this metalogical square is when we consider the situation for propositions alone (when the theory is empty):



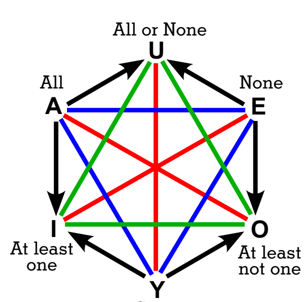
In this square does not appear contingent propositions, the only real propositions according to Wittgenstein. And at the level of the metalogical square for consequence does not appear the notion of independence, which is also a crucial notion in modern logic in particular in view of Gödel's incompleteness theorem. This problem can be solved by using the hexagon of Robert Blanché (1966).

4 The metalogical hexagon of consequence

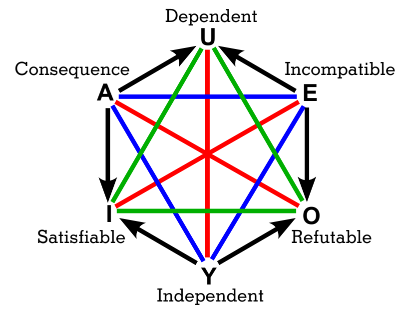
The hexagon of Blanché is the following structure, combining a triangle of contrariety and a triangle of subcontariety giving rise to a hexagon in which we find back the square:



A particular case of the hexagon is the hexagon of quantification that resolves several problems that appears in the square of quantification (for a detailed discussion about this, see Beziau 2012a):

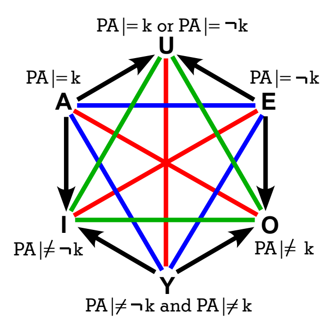


This hexagon can be used to describe in a nice structural way the relations between many notions. Here is the hexagon for consequence:



This metalogical hexagon describes perfectly the six positions of a proposition vis-à-vis a theory. It has a universal character, since it is an abstract structure that can be used for the development of many particular logical systems. It can be applied to any consequence relation, even if no axioms are given for it, in the spirit of the axiomatic emptiness of universal logic.

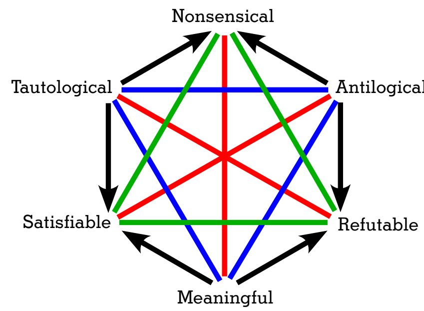
The use of such a coloured diagram is very useful to understand in a direct, quick and synthetic way basic notions of modern logic, corresponding to the notion of *Übersichtlichkeit* that Wittgenstein was found of (see e.g. Sin, 1980). Here is a particular version of this hexagon for the case of Peano arithmetic (PA):



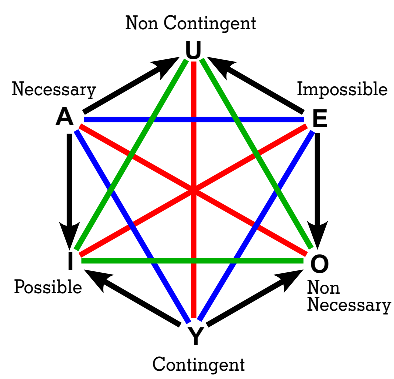
In this diagram the consequence relation $\models$ can be interpreted in various ways. It can be the proof-theoretical notion of classical logic, but it can also be the model-theoretical notion of this same logic. We can also consider that the logic is not classical logic, but intuitionistic logic, or another non-classical logic. One important question is to know for exactly which logics this diagram about Peano arithmetic holds.

5 The metalogical hexagon and the hexagon of modalities

In the case of the metalogicalhexagon of propositions, how can we call the U-position? We can say that a proposition which is either tautological or antilogical is nonsensical. In this case it is good to say that a proposition which is in the position Y, in contradictory opposition to U, is meaningful, following Wittgenstein's original idea. We have then the following picture:



In our metalogical triangle of contrariety, we used the word contingencies to speak about propositions which are in the Y-position. If we want to keep this terminology, then the U position can simply be called non-contingent like in the metalogical hexagon of modalities:



We can in fact use this modal hexagon as a metalogical hexagon, saying that a tautology is a necessary proposition, an antilogy an impossible proposition, a satisfiable proposition, a possible proposition, a refutable proposition, a non-necessary proposition.

This is a perspective according to which modalities are metalogical, an idea promoted by Wittgenstein (*Tractatus* 4.464). But Wittgenstein defended also the idea that modalities are strictly metalogical, that they cannot be considered at the same level as connectives (*Tractatus* 5.525). A teenager acquainted to S5 and other modal logics much popular nowadays can consider this claim as a weakeness of Wittgenstein, which is often much criticized by hardcore logicians. But this can be interpreted in fact as a firm distinction between logic and metalogic, a distinction which took time to be established.

Tarski (1936) claimed that this distinction, so much important for Gödel's theorem, was established in the Polish school and that it was made clear to Gödel, only after he talked with him in a discussion he had with him in Vienna. What we can maybe say is that Wittgenstein was against the mixture of the two levels which is a crucial step in Gödel's theorem, in particular through the arithmetization of syntax, and which manifests also through the logic of provability which has been developed to give a precise account of Gödel's incompleteness theorems.

And the interaction between logic and metalogic appears in different manners with modalities, Lukasiewicz for example was considering possibility as a truth-value. Considering the above logical hexagon of modalities, it is possible to develop different many-valued logics.

6 The metalogical hexagon and the semiotic hexagon

The metalogical hexagon of oppositions for propositions or for consequence is a very simple and useful tool. It clarifies and gives a direct understanding of fundamental logical notions and as we have emphasized it has a universal character.

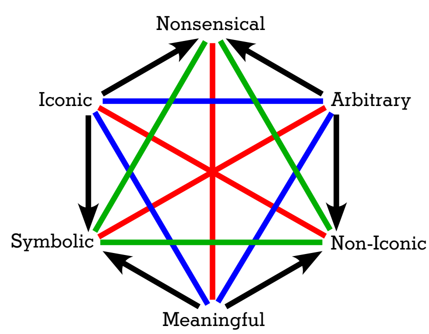
One metalogical hexagon we have presented involved notions as *meaning* and *nonsense* for propositions. Such notions are defined through the idea of truth. If we go to general semiotics, the situation is quite different. In general it is not relevant to say that a sign (or group of signs) is true or not. A more relevant question is the relation between the sign and what it is supposed to designate.

Some signs are arbitrary in the sense that they don't have a particular relation with what they are supposed to express. This is typical of alphabetical language: the word "cat" has no relation with the reality of the animal. There is in this case no connection between the signifier and the signified to use the terminology put forward by Saussure who was the main promoter of the idea of arbitrary sign by opposition to symbols. Saussure by doing this distinction was using the word "symbol" in its original Greek sense.

The word "symbol" has been used in different ways. In modern logic, following the formalist ideology, symbols are meaningless signs. A different meaning, closer to the original, is the one given by Peirce, who was making the distinction between symbols and icons. For Peirce, an icon is a sign having a strong connection to reality, in the case of symbols there is for Peirce a relation between the sign and reality but this relation is conventional. Sometimes it is difficult to make the distinction between the two, but we can say that a pictogramatic language as the one promoted by Neurath is far to be purely conventional. Symbolism may have two dimensions: the balance is symbolically representing the idea of justice and this symbol of justice is represented by a sign which is a picture of a balance.

In both cases this is not a pure convention.

The opposition between arbitrary signs and symbols can be seen as contradictory. A symbol can be seen as a particular case of icon, but we can say that an icon, a representation of reality such as a photograph is quite meaningless, in the sense that it does not articulate meaning but just reproduce reality. We then have the following hexagon:



The hexagon itself, and other diagrams, like Venn's diagrams, stand in the Y position. It is a highly meaningful sign, something which is not arbitrary (it is possible to express the same thing with sequence of arbitrary signs, this would be much lengthier and difficult to grasp) and not iconic (this is not a picture of reality like a photograph of the statue of liberty).

The use of such a diagram helps to develop logic in a truly symbolic way (see Moktefi and Shin 2013 for a recent account about diagrams in logic).

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