ARGUMENTOS
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Tiago Tranjan*

Mathematics and essence

ABSTRACT

Arithmetic seems to provide philosophers and mathematicians with the typical domain where it makes sense to talk about "essences", "essential properties" and "necessary propositions". In this paper I try to examine the subject, following the lines of investigation suggested by Wittgenstein from his middle period on. Beginning with a specific question – Is it part of the essence of the number 3 to be followed by the number 4? –, I intend to examine different aspects of the problem, with three main goals at sight: 1) Understanding more thoroughly the role and limits of formal analysis in philosophical questions, particularly those related to the philosophy of mathematics; 2) Correctly locating the note of "essentiality" and/or "necessity" commonly associated with arithmetic; and 3) Suggesting a general way of understanding the idea of essence and necessity, in its strict connection with human language and experience.

Keywords: Philosophy of mathematics; Mathematical necessity; Counting systems; Linguistic necessity; Wittgenstein on mathematics.

RESUMO

A aritmética parece fornecer a filósofos e matemáticos o típico domínio em que faz sentido falar de "essências", "propriedades essenciais" e "proposições necessárias". Neste artigo eu tento examinar esse tema, seguindo as linhas de investigação abertas por Wittgenstein a partir de seu período intermediário. Partindo de uma questão específica – Pertence à essência do número 3 ser seguido pelo número 4? –, eu abordo diferentes aspectos do problema, com três objetivos centrais: 1) Compreender mais detalhadamente o papel e os limites da análise formal em questões filosóficas, particularmente aquelas ligadas à filosofia da matemática; 2) Situar corretamente a nota de "essencialidade" e/ou "necessidade" comumente associada à aritmética; e 3) Sugerir uma maneira geral de compreender a ideia de essência e necessidade, em sua estrita ligação com a linguagem e a experiência humana.

Palavras-chave: Filosofia da matemática; Necessidade matemática; Sistemas de contagem; Necessidade linguística; Wittgenstein; matemática.

* Doutor e Professor de Filosofia da UNIFESP. Email: ttranjan@hotmail.com

In this article I will try to pursue, along wittgensteinian lines, the following question:

(1) Is it part of the essence of the number 3 to be followed by the number 4?

My hope is that the analysis of this case will advance our understanding about an important issue, not only in philosophy of mathematics, but for philosophy in general.

One initial observation may help avoid unnecessary queries regarding my procedure. In order better to grasp our starting question, I will recast it in many different forms, the exact synonymy of which may be disputed. This shouldn't be a problem. From the point of view I am adopting, there is no such thing as "exact synonymy". On the contrary, it is precisely the multi-shaped emergence of the above question, in approximately synonymous forms, which most interests us.

In the first place, someone may not be at ease with talk about "essences", and all its platonic resonances. So we may try to substitute the idea of "essential property" for that of "essence", hoping that the new notion is more apt to a consistent treatment. Accordingly, we may reformulate our question in this somewhat more flexible form:

(2) Is it an essential property of the number 3 that of being followed by the number 4?

There are, of course, different ways of understanding the general idea of "essential property" here present. More precisely, there are different suggestions about how to specify the content of this idea. The most traditional approach is what we may call the "modal approach". According to it, a certain property is essential to an object if, and only if, the attribution of that property to the object is necessary. In a technical fashion, we could say that a property P is essential to the object o if, and only if, "P(o)" is a necessary proposition.

In spite of some problems raised by this approach, thoroughly discussed in recent literature, I will accept it as the most adequate one. It has what I consider to be a definite advantage: it manages to make the clearest transition from a talk about objects and their properties to a talk about propositions. We no longer have to muse over the attribution of a property to an object – after all, what are objects? What are properties?-, we only have to consider the status of a certain proposition as "true", "necessarily true" etc. Why I consider this to be a definite advantage will become apparent in the sequence of this paper.

If we apply the modal approach to our question, we arrive at the following formulation:

(3) Is the proposition "The number 3 is followed by the number 4" a necessary proposition?

How is this question to be understood? It is part and parcel of the modal approach to explain it in terms of possible worlds. We finally get:

(4) Is the proposition "The number 3 is followed by the number 4" true in every possible world?

We could now tackle the question following two distinct strategies. One of them is the formal strategy, typical of modern logic. The driving force behind it is this: In order to give substance to the analysis, we elaborate a symbolic theory, with mathematical rigour, capable of giving more precise articulation to key notions such as "possible world" and "being true in a possible world". Such strategy seems particularly promising when dealing with mathematical propositions. We will have much to say about it later in this paper. Before doing so, however, it will be most valuable to follow the second strategy, and look at the situation from the informal point of view – or, should I say, the non-formal point of view. After all, the modal approach antedates modern logic and its formal rigour by some centuries, and it is only fair to see where it can lead us. It will also be a very good guide to the formal analysis.

Exploring question (4) non-formally means that we will not come up with some elaborate symbolic theory, capable of giving mathematical structure to the investigation. Rather, we will try to answer question (4) somewhat more directly, based on some non-formal grasp of the notions involved. Once again, different ways of (non-formally) grasping these notions could be suggested, and much discussion could be raised about the problems therein involved. What, after all, should be considered a "possible world"? How are we to evaluate propositions in them?

Fortunately, here, we may cut the discussion short. We are already close enough to ground. As regards basic arithmetical questions, such as the one we have at hand, any argument would not amount to much. Conceptions may vary at will, but the general answer to our question will always result affirmative: in every possible world, so we are bound to conclude, the number 3 is followed by the number 4. Why am I saying that?

On close inspection, it is not difficult to understand what is going on. The answers to question (4) will be convergent because, having adopted the modal approach, nothing really modal is at stake here. The number 3 will be followed by the number 4 in every possible world, not on account of some common property of the possible worlds, but on account of the common way we will count objects on them. The whole analysis will fall prey to this simple fact: In every possible world, when it comes to our counting their objects, we make the

4 succeed the 3. Nothing is said about the configuration of different possible worlds, only about the common way used to describe them (more precisely: the way used to count objects on them).

Has the non-formal modal analysis turned out to be a failure? Yes and no. It has revealed us something important, to which I have been trying to direct our attention. But it certainly did not deliver what it promised. Let us compare the simple arithmetical question we have been discussing with this other question: Does the proton attract the electron in every possible world? The answers here may vary. Namely, they will depend upon a specification of what counts as possible worlds. This is the very purpose of a modal analysis: to explain the (necessary) truth of propositions in terms of the way things are or could be; in terms, that is to say, of some conception about possible worlds. But we are unable to reach this point when dealing with the simple arithmetical question we have at hand. The modal analysis, at this level, gives us no information or insight about why "the number 3 is followed by the number 4" is a necessary truth. We are just forced back to the starting point: This is the way we count objects; but does it carry any note of necessity?

Now we may try to follow the formal strategy, and check if it can deliver us something more substantial. This is a common – and in some sense justified – hope. After all, formal analysis can offer some very interesting results about mathematical structures and configurations. The hope is that, by treating numbers in a more rigorous fashion – for example, by offering precise definitions of the individual numbers, as well as of the general notion of number – it might reveal the facts which are involved, or which are behind, or which are expressed, in their mutual relations. By now, however, the fear should be unavoidable that even the formal analysis will be exposed to some kind of short-circuit, just as happened to the non-formal analysis.

The problem at hand is a delicate one, for it demands compliancy with two opposite requirements. On the one hand, it calls forth a very general discussion about what can and what cannot be delivered by formal analysis as a method. In fact, many such analyses are possible, and what is in question is not the adequacy of one proposed route or another. What is in question is the possibility of getting any further – of getting new grounds for understanding the situation – by any formal means available. This is clearly a generality requirement.

On the other hand, however, the correct evaluation of a formal result can only be achieved through a careful consideration of the particular case at hand. This is an essential point to be understood. Unfortunately, it is also a point only too easily forgotten. What must be investigate dare always the connections which can be established among the formulas of the particular calculus at hand, the formulas of this calculus and of other calculi, and the formulas of this calculus and certain domains of natural language. Generality will be always lacking, with respect to the formal method, because there is no hidden "reality" behind different formalisms to naturally unify them; there is no common "subject matter" whose "correct description" would serve as a shared parameter. In fact, formal "theories" do not aim at "seeing" (and describing) anything. They normatively establish a way of manipulating – transforming and combining – a particular vocabulary (set of symbols).And though relations can obviously arise between different formalisms, they too have to be normatively established.The task, in any case, will always be the same: a detailed analysis of the many symbolic transformations made possible by a *specific* calculus.

Here we have come to a halt. Because of the very general nature of our inquiry, we should look for some very general approach. What we want to know is if *any* formal analysis can help us to settle question (4). However, as I briefly argued, there is no such thing as a "general theory" of formal theories, any such an attempt necessarily resulting in some new particular calculus, showing how to formally connect other previously available calculi.

In what follows, I will not try to settle question (4) in a definite fashion. Whether it can or cannot be settled by general arguments, and what sort of arguments should then be used, will remain as an open problem, to be tackled in another occasion. I will do something different: I will examine a particular case of formal analysis – actually, a whole family of particular cases –, hoping that the suggested procedure has some sort of general value. It may have such general value, not by giving a general description of the "formal realm" (its possibilities and impossibilities), but by showing, with respect to the question at hand, a general way of understating the results of formal analysis. To put it briefly: I will discuss a very representative case of formal analysis, and show that it leads to the same kind of short-circuit we met with when dealing with non-formal analysis. By doing so, I will uncover a general pattern, pointing to some very general problems in the interpretation of formal results. This is still no general result – in the rigorous sense afforded by formal analysis itself –, but now the burden of proof is reversed. If someone believes he can forestall the short-circuit, it is up to him to set up his formal trap, after carefully taking into consideration the dangers I draw attention to.

Since we are dealing with the modal approach, what will be required of a formal analysis is a formal semantics. It will be necessary to specify a class of models (the formal correspondent of "possible worlds"), and to give a truth definition for sentences with respect to the models of this class. The class of relevant sentences, too, must be specified, for the truth definition will not apply to any sentence of any language, but only to the sentences of some reasonably specified formal language. All this can be done, of course, in many different ways.

Let us begin by the one necessary starting point, the sentence whose truth in every possible world (every model) is to be evaluated: "the number 3 is followed by the number 4". This sentence must be translated into a formal sentence, of the kind to which the truth definition applies. The criteria of adequacy for such a translation are not given beforehand. Once again, there is great confusion prevailing here, due in most part to a misconception about the nature of the formal method and what it can deliver. Many logicians believe they can concoct a formula which, when interpreted (or, may be, "correctly" interpreted), expresses some facts; and that precisely these facts are (or so they propose to regard them) what is behind the assertion that the number 3 is followed by the number 4. Therefore – because some facts could be discerned that supposedly serve to clarify our use of numbers, and because these facts can be rigorously expressed by a formula in the formal language –, this formula should be seen as an adequate formal translation of the sentence in question. Proving that the facts expressed by such a formula obtain in every possible world – that is: that the formula holds true in every model – would be a good way, probably the best possible way, to prove that "the number 3 is followed by the number 4" is a necessary proposition, and that it is part of the essence of the number 3 to be followed by the number 4.

But that will never do. That is not what the formal analysis can offer. Formal semantics will establish some formal relations between a mathematical structure, the models, and other formal-symbolic structures, the formal sentences. The establishing of such relations can be extremely interesting from the mathematical point of view. They may allow new connections and transformations to show up within the symbolism. But they cannot express and prove facts which justify the necessity of some proposition.

To see this, we shall consider, as I promised, a typical example of a translation for sentence (4). First, I will give the general scheme of the proposed translation. It should point to some facts that, purportedly, lie behind the mathematical assertion. But the scheme is still no formal sentence in the desired sense, and cannot be the end of the analysis. So we will have to move a step forward in the direction of formalization. Just one step, an altogether indispensable step. But in doing so we will bring forth the central difficulty the formal method has to deal with.

The general scheme is this:

$$\text{(A)} \qquad \forall P\, \forall Q \begin{cases} \textit{There are } 3\ P \\ \wedge \\ \textit{There is } 1\ Q \\ \wedge \\ \textit{There are no} (P \wedge Q) \end{cases} \rightarrow \qquad \textit{There are } 4\ (P \vee Q)$$

It should be clear why the scheme cannot be left in the above form. If we are trying to analyse the relation between the numbers 3 and 4 in terms of universal facts obtaining in every possible world, and not as a *de facto* question about how we count in our world (how we use the words "three" and "four"), we cannot use the numbers themselves in the translation. This much is widely recognized. Usually, the hope of doing away with numbers as primitive undefined symbols assumes a quite distinct direction: Variables are used to express the fundamental facts behind numeric relations. So let us go one step more formal, and use variables. We should get something like this[1]:

$$\text{(B)} \qquad \forall P\, \forall Q \begin{cases} \exists\, x, y, z \qquad P \\ \qquad\qquad \wedge \\ \exists\, w \qquad\qquad Q \\ \qquad\qquad \wedge \\ \sim\exists\, x' \quad (P \wedge Q) \end{cases} \qquad \rightarrow \qquad \exists\, x, y, z, w \quad (P \vee Q)$$

Now, how are we to understand this sentence? The important point, usually missed, is the necessity we have of counting the variables. This sentence, which came up as the best hope we have of explaining the necessity of an arithmetical statement, can only do its job after we have counted the variables as being three in the first line within the brackets, and as being four after the implication sign.

To see this more clearly, we put forward some small notational variations of the same formal sentence (or formal scheme), all in perfect accord with widespread logical practice:

$$\text{(C)} \qquad \forall P\, \forall Q \begin{cases} \exists\, x, y, z \qquad P \\ \qquad\qquad \wedge \\ \exists\, x \qquad\qquad Q \\ \qquad\qquad \wedge \\ \sim\exists\, x \quad (P \wedge Q) \end{cases} \qquad \rightarrow \qquad \exists\, x, y, z, w \quad (P \vee Q)$$

$$\text{(D)} \qquad \forall P\, \forall Q \begin{cases} \exists\, x, y, z \qquad P \\ \qquad\qquad \wedge \\ \exists\, w \qquad\qquad Q \\ \qquad\qquad \wedge \\ \sim\exists\, x' \quad (P \wedge Q) \end{cases} \qquad \rightarrow \qquad \exists\, a, m, l, e \quad (P \vee Q)$$

In (B), the use of variables was made particularly easy – and therefore particularly deceiving – because of two tricks ("tricks" being literally what they are): the *non-repetition* of variables in the vertical lines within the brackets; and the *repetition*, after the implication sign, of the variables already used in the brackets (not to mention the use of variables in alphabetical sequence). Cases (C) and (D), which avoid those tricks, highlight the necessity of correctly counting the variables.

[1] I will not set up, in detail, the vocabulary and grammar of some specific formal language. I will give the general outline of the formalization, in such a way that any competent logician can put it in a more strict form, if the he wants to do so.

Case (E) below is where we cut directly through this difficulty, using number-indexed variables:

$$\text{(E)}\qquad \forall P\,\forall Q \begin{cases} \exists\, x_1, x_2, x_3 & P \\ \qquad\quad \wedge & \\ \exists\, x_1 & Q \\ \qquad\quad \wedge & \\ \sim\exists\, x' & (P \wedge Q) \end{cases} \qquad \rightarrow \qquad \exists\, x_1, x_2, x_3, x_4 \quad (P \vee Q)$$

Depending on the way we understand the indexes, however, we are clearly cheating. We can abstract their normal use as devices for counting, and think of them as dumb, sequence-neutral names. This would be reasonably fair; unfortunately, it would put us right back to where we were before. Alternatively, we can grant the indexes their normal use, in the well-known sequenced structured. But that would be just counting the variables in the usual way. It is not what we want, since our goal is precisely to justify the necessity of our counting.

No doubt we can set up some more elaborate tricks to solve our problem. One of them is particularly important, because of the foundational pretensions it is often associated with: the procedure of checking for a one-to-one correspondence – in this case, between the variables within the brackets and the variables at the right-hand side of the implication sign. With respect to the examples above, we can materialize this procedure by means of the following juxtaposition-schemes:

$$\text{(B')}\quad \begin{array}{l} \mathrm{x\ y\ z} \\ \underline{\qquad\quad \mathrm{w}} \\ \mathrm{x\ y\ z\ w} \end{array} \qquad\qquad \text{(C')}\quad \begin{array}{l} \mathrm{x\ y\ z} \\ \underline{\qquad\quad \mathrm{x}} \\ \mathrm{x\ y\ z\ w} \end{array}$$

$$\text{(D')}\quad \begin{array}{l} \mathrm{x\ y\ z} \\ \underline{\qquad\quad \mathrm{w}} \\ \mathrm{a\ m\ l\ e} \end{array} \qquad\qquad \text{(E')}\quad \begin{array}{l} \mathrm{x_1\ x_2\ x_3} \\ \underline{\qquad\qquad \mathrm{x_1}} \\ \mathrm{x_1\ x_2\ x_3\ x_4} \end{array}$$

This kind of scheme is certainly sufficient, at least in the simplest cases (arithmetic of small numbers), to solve one of the problems we have at hand: that of finding the correct amount of variables to be put after the implication sign (but imagine having to check in this manner the formal expression for a sum of two nine-digit numbers).However, this is not enough, for there is a second problem. One thing is to check for the one-to-one correspondence between juxtaposed rows. This we can do in any (simple) case, such as (C'). But (C') corresponds to just that: a checking of variables to see if the formal sentence was correctly set up.We are looking for something subtly different. We are looking, not just for a formal sentence correctly set up (one that will turn out to be true when interpreted), but for the formal rendering of a particular sentence in natural language: "the number 3 is followed by the number 4". Such a goal implies a second task, not accomplished by (B), (C) or (D), though they are all correctly set up. It implies the task of correlating a particular formal sentence (supposing it correctly set up) with the particular sentence it is supposed to be the translation of.

This is no idle task. It shows us that the act of gathering things and checking their one-to-one correspondence is not equivalent to the act of counting those same things. Let us take a look at a revised version of (E'), where I use only the indexes:

$$\text{(E'')}\quad \begin{array}{r} 1\ 2\ 3\ \\ \underline{1} \\ 1\ 2\ 3\ 4 \end{array}$$

In a way, the very construction of this scheme – and its use to *check* for a one-to-one correspondence – shows us something mighty important. The sequence "1 2 3", followed by "1", is still not the sequence "1 2 3 4". It is still just "1 2 3 1"! The last line of (E') or (E'') –where indexes are used in the usual way as counting numbers, *all the way up to 4*, and not as just another set of dumb names –, is something quite new. This line is not the same as "x y z w", "a m l e" or whatever we could put there to mark positions. Nor are we merely playing with names, as if "four" could not be called "vier", or the symbol "e" could not be used instead of "4". What is new in the last line of (E') and (E'') is not the name "4", but the use of a novel ordered structure, beyond the one already available (supposing "1 2 3" was already available), in which "4" comes after "3".

To make this difficult point more explicit, it is worthwhile to consider the same kind of formal rendering for a slightly different sentence –a very simple arithmetical proposition, such as "five plus seven equals twelve". We may try something like this:

$$\text{(F)}\qquad \forall P\,\forall Q \begin{cases} \exists\, a,b,c,d,e \qquad P \\ \qquad\quad \wedge \\ \exists\, f,g,h,i,j,k,l \quad Q \\ \qquad\quad \wedge \\ \sim\exists\, x \qquad (P\wedge Q) \end{cases} \rightarrow\ \exists\, a,b,c,\ldots,k,l \quad (P\vee Q)$$

Or:

$$\text{(G)}\qquad \forall P\,\forall Q \begin{cases} \exists\, a,b,c,d,e \qquad P \\ \qquad\quad \wedge \\ \exists\, f,g,h,i,j,k,l \quad Q \\ \qquad\quad \wedge \\ \sim\exists\, x \qquad (P\wedge Q) \end{cases} \rightarrow\ \exists\, m,n,o,p,q,r,s,t,u,v,x,z \quad (P\vee Q)$$

Once again, we should distinguish two distinct but related problems. First we have the problem of correctly setting up the formula, that is to say, knowing how to put variables after the implication sign. Second we have the problem of correlating a particular formula (supposing it was correctly set up) with the particular sum it is supposed to translate. The first problem is still not distinctively a "how-many" type of problem. Its solution is possible by means of juxtaposition schemes, which dispense with a "how-many" type of answer:

$$(F') \quad \frac{\begin{array}{l}\text{a b c d e} \\ \qquad\qquad \text{f g h i j k l}\end{array}}{\text{a b c d e f g h i j k l}} \qquad\qquad (G') \quad \frac{\begin{array}{l}\text{a b c d e} \\ \qquad\qquad \text{f g h i j k l}\end{array}}{\text{m n o p q r s t u v x z}}$$

The second problem, on the contrary, clearly puts the question: How many variables are there in the different positions of the formula? For this problem, both schemes fall very short of giving an answer. Once again, the fact that a sequence of variables labeled "a b c d e" and a sequence of variables labeled "f g h i j k l" can be put into one-to-one correspondence with the sequence of variables labeled "m n o p q r s t u v x z" or "a b c d e f g h i j k l" is something very different from the sum "five plus seven equals twelve". In order to get to this sum, I would have to know – and that means *to count* – how many variables I have in "a b c d e"; how many variables I have in "f g h i j k l"; and how many variables – this is crucial – I have in "m n o p q r s t u v x z" or "a b c d e f g h i j k l". Only after that much counting I would be in a position to say that sentence (F), or (G), is the desired formal expression of that particular sum.

Let us see what happens, in this case, with formalizations in which numbers are used as indexes for the variables:

$$(H) \qquad \forall P\, \forall Q \begin{cases} \exists\, x_1, x_2, x_3, x_4, x_5 \qquad\qquad P \\ \qquad\qquad\qquad \wedge \\ \exists\, y_1, y_2, y_3, y_4, y_5, y_6, y_7 \quad\; Q \\ \qquad\qquad\qquad \wedge \\ \sim\exists\, x \qquad\qquad\qquad\quad (P \wedge Q) \end{cases} \rightarrow \quad \exists\, x_1, \dots, x_5, y_1, \dots, y_7 \quad (P \vee Q)$$

As expected, schemes (H') and (H'')below, if they are necessary at all, can solve our first problem:

$$(H') \quad \frac{\begin{array}{l} x_1\, x_2\, x_3\, x_4\, x_5 \\ \qquad\qquad\quad y_1\, y_2\, y_3\, y_4\, y_5\, y_6\, y_7 \end{array}}{x_1\, x_2\, x_3\, x_4\, x_5 y_1\, y_2\, y_3\, y_4\, y_5\, y_6\, y_7}$$

$$(H'') \quad \frac{\begin{array}{l} 1\,2\,3\,4\,5 \\ \qquad\quad 1\,2\,3\,4\,5\,6\,7 \end{array}}{1\,2\,3\,4\,5\,1\,2\,3\,4\,5\,6\,7}$$

Here, however, it becomes particularly easy to realize that such schemes, though enough to solve our first problem, do not solve our second problem. In the first line we count up to 5. In the second line we count up to 7. Putting them together, what do we get? A counting of five, followed by a counting of seven – but still not a counting of twelve! No matter how many variations are proposed for this scheme, no matter how many different names are given to the variables, each time checking for their one-to-one correspondence by means of juxtaposition-schemes, none but one sequence will deliver the number twelve as a word occupying a specific position within an ordered sequence of words. This is, of course, the sequence "1 2 3 4 5 6 7 8 9 10 11 12". In the end, the scheme will have to look like this:

(H''') 1 2 3 4 5
1 2 3 4 5 6 7
1 2 3 4 5 1 2 3 4 5 6 7
1 2 3 4 5 6 7 8 9 10 11 12

Incidentally, this is the very reason why Kant correctly considers that the concept of twelve is not reducible to the concept of five, the concept of seven, and the concept of sum. In Kantian terminology, the number twelve corresponds to a new synthesis, not given by the successive synthesis of the numbers five and seven, and which must be performed on its own.

Another way of putting this result is to say that the act of comparing sets on a one-to-one basis and the act of counting are fundamentally distinct: they possess different structures and different possibilities. The connection between them is as deep as it is complex, and should be examined in detail. It may vary for different contexts. One thing, however, results clear from the analysis above: the act of counting is not reducible to the act of correlating sets[2]. And when we inquire into the nature of sentences like "the number 3 is

[2] Of course we could describe the act of counting as the act of making a one-to-one correlation between a determinate set of objects – say, the variables in a formula – and the set of natural numbers. In a certain sense, this seems to be a perfectly correct description of the situation. But then we should realize this is not the same as saying that the act of counting is reducible, or equivalent, to the act of establishing one-to-one correlations in general. What is relevant, in the act of counting, is that the correlation is established with a very particular ordered structure, the natural numbers, which must be available before the correlation is made. This is why the number 4, coming *after* the number 3,is something new. This is why the number 6 billion, when applied to the inhabitants of Earth, does a very particular job: it is not just the label I associate with the last inhabitant in a pointless one-to-ne correlation with dumb names. In fact, I could (could I really?) establish a one-to-one correlation between the inhabitants of Earth and some set of dots in a gigantic sheet of paper, or some set of names listed in many sheets of papers (this is what civil registers are supposed to do), but that would not be *counting* the inhabitants of Earth. When such correlations were finished, I would still not know *how many* inhabitants there are on Earth. In the case of the list of names, I would know to name each individual person; in the case of the dots, I would simply not know what to do with it. But when I say there are 6 billion inhabitants on the face of Earth, the number 6 billion carries a very precise information, because 6 billion is a natural number, with a particular position in the ordered structure of the natural numbers.

followed by the number 4", we are certainly inquiring into the structure of the act of counting.

Going back to our original concern, which is the formal analysis of sentence (4), we are now in a good position to appreciate how close we got to the situation we met with in the non-formal analysis of the same sentence. Once again, we find ourselves unable to disclose facts which lie behind the necessity of the sentence "the number 3 is followed by the number 4". Once again, the only thing we seem able to do is to take notice of the particular way of counting we use. This time, the way we count variables, with four variables coming after three variables.

What becomes progressively clear, as we deepen our investigations in the direction suggested, is this: At the basic arithmetical level, it is not the formal analysis which explains the necessity of "the number 3 is followed by the number 4". It is rather the opposite. It is the use of arithmetical language, with all its structure already available, which is revealed to be fundamental, at least in most cases, to the possibility of formal analysis.

Proceeding in this direction, however, we can do still better. We can reverse our expectations and consider an altogether different situation: a specific linguistic context where the number 3 is *not* followed by 4. Is this possible? It certainly is. For we can imagine a linguistic community – and anthropologists may occasionally exhibit them –whose numbering system is as follows:

1, 2, 3, many.

When I say we can "imagine" this community, I have in mind something very precise. We can describe the activities of such a community, and a linguistic practice coherent with those activities. This community would have a very determinate practice of counting. And this practice would not collapse at any point, say, by the lack of criteria to use the numbers, by inconsistencies hidden in the system, or by the felt absence of other "greater" numbers. To the question "How many parents do you have?", the answer would be: 2. To the question "How many fingers do you have?", the answer would be: many. This community would even be likely to develop its own arithmetic, wherein"2 + 1 = 3" and "2 + 3 = many" would be true propositions. (For them, "many - 2" would be an indeterminate expression, such as"8/0" in our school arithmetic, or such as "3 - 5" in 12^{th} century arithmetic).

It is here, I should notice, that the linguistic approach reveals its advantages. When considering, as we now do, the possibility of a numbering system altogether different from the systems we are familiar with, we have at hands something very definite to exhibit and examine: a coherent linguistic practice, which is public and can be described. (I am tempted to say this is all we have to exhibit and examine, that is to say, that is all we have to go on from).

With respect to the "1, 2, 3, many" language, two points need particular stressing. Firstly, there is nothing "wrong" with the language. It does not misfire in anyway whatsoever. It can be put to use, according to shared rules; it can be learned by foreigners, just as any other language. Secondly, there is nothing amiss with the language, no gap waiting to be filled. To see this, we can compare this language with the situation of modern European languages (modern English, for instance) before the number 0 made its appearance in it, or before – for this took sometime –the number 0 obtained general acceptance. Certainly, after being incorporated to our language, the number 0 did and does a great job. It is useful in many situations, and permits many new operations and transformations. But English was perfectly all right before that. English was not lacking something, and the number 0 could just as well have never been invented or come into use. English was also not lacking the "imaginary" numbers before the 18th century, as it is not lacking a word for what Portuguese-speaking people call "saudades". (Would such a feeling be missing in English-speaking people? Or would they have this feeling, but are still in search for an adequate word?)

Of course, we cannot but have the feeling that something – namely the number 4 – is missing in this imaginary community we are investigating. Maybe this impression is related to the idea, completely false, that people in this community would be lacking some capacity, for instance, the capacity to distinguish more than three objects, or to correctly taking them into account. This is absolutely not the case. They have this capacity all right (as they may have any other capacities we have). They just count differently. And they may express their capacity to distinguish more than three objects, not by counting, but in many other ways, the most obvious being the use of names. Someone could say, "I have many brothers: John, Lucas, Mathew, Mark and George". He would certainly distinguish each of them, and would know that George was missing, even though he would also count "many" brothers with just John, Lucas, Mathew and Mark present.

We have reached, now, a strange situation. The last step in our analysis seems to show that "being followed by 4" is not an essential property of 3; that "the number 3 is followed by the number 4" is not a necessary truth. On the other hand, we have the distinct impression that, according to our conceptualization, being followed by the number 4 is absolutely essential to the number 3. Moreover, this impression is corroborated by most philosophical analysis on the subject, be them formal or not[3], be them modal or not[4].

[3] Mathematics has traditionally supplied Philosophy with most of its examples of "essential properties" and "necessary truths"; and most model-theoretic semantics "prove" arithmetical sentences to be necessary truths.
[4] According to other, non-modal approaches, being followed by 4 would be essential to the number 3 because, for instance, it is part of the definition of 3 (definitional approach); or because it explains the other properties possessed by 3 (explanatory approach).

So it seems we have been brought to some kind of *aporia*.

As happens with any other *aporia*, however, thistoo must be revealed as only apparent. The solution to our puzzle, I would like to argue, depends on realizing (and clarifying) the compatibility between these two assertions:

- It is essential, in the English language, that the number 3 be followed by the number 4.
- It is *not* essential to the English language that the number 3 be followed by the number 4. (Or: it is contingent to the English language that the number three be followed by the number 4).

For those who prefer to talk about the necessity of propositions, the pair becomes:

- "3 is followed by 4" is a necessary proposition of the English language.
- " '3 is followed by 4' is a necessary proposition of the English language" is contingent.

This apparently strange compatibility is the result of a simple situation, only too often disregarded: The English language itself is contingent; but, within it, many things are not contingent. To put it another way: The fact that the English language is contingent does not imply that all its propositions are contingent, nor prevents the development, within it, of a manageable concept of necessity.

That the English language is contingent is reasonably obvious. All human languages are contingent, just as is contingent human life and the brain which allows humans to use their different languages. Such a contingency holds also for the mathematical portions of English, including basic arithmetic, despite all its appearance of eternity. The English language (or what came to be the English language, as we know it today) could have never developed a counting system, or could have developed a different counting system and a different arithmetic from those we know today. The number 4, as well as the number 3 – and of course we are not talking about the specific sound of these words –,could have never made their appearance in it. In this sense they are contingent to the English language.

However, within the English language, as it is spoken and understood by us, it is absolutely essential to the number 3 to be followed by the number 4. The grammar of the number 3 involves its coming after the number 2 and before the number 4, as part of a very definite counting system, endowed with very precise rules. This is why all our analysis, formal or not, carried within the boundaries of the English language, are bound to conclude for the essentiality of properties of the number 3, and for the necessity of propositions involving them. It is within English, a contingent language, that the essence of 3 is born and resides.

Here there is a lesson which extends far beyond the immediate subject we have been considering. Up to this point, our discussion was based on the unraveling of a specifically mathematical-philosophical question. However, the mathematical discussion points to a much wider situation. I am referring to the need of realizing – and philosophically clarifying – the compatibility between these two very general assertions:

- Every essence is within language (within some language).
- There is nothing essential to language.

Or yet:

- Every necessity is within language (within some language).
- Every language is contingent.

The creation of essence and necessity from sheer contingency is the distinctive mark of human, linguistic, experience. In a certain sense, it *is* language itself. The understanding of this point, as Wittgenstein well realized, establishes a completely new philosophical agenda, affording a way of superseding many age-old insoluble problems.

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